Mathematical Appendix to 'Progressive Services…'

(For convenience of reference, equation numbers in this Mathematical Appendix follow from those in the Appendix in the article.)

A. Proof of Proposition 2

Proceeding backwards and up the saddle-path in the final Regime 1 (Fig 1), either (a) $\dot{n}_{S1}/n_{S1}$ will fall to 0 while $\dot{n}_{M1}/n_{M1}$ remains at 0, or (b) $\dot{n}_{M1}/n_{M1}$ becomes positive while $\dot{n}_{S1}/n_{S1}$ is still positive. We consider first the latter possibility, which implies that Regime 1 has, proceeding backwards, given way to Regime 2 in Appendix A.1. We also note an important preliminary fact. At the transition point between Regimes 1 and 2, $\dot{G}/G$ (= $-(\dot{v}_{S1}/v_{S1}+\dot{n}_{S1}/n_{S1})$ = $-(\dot{v}_2/v_2+\dot{n}_2/n_2)$, with all $\dot{v}_k/v_k$ being continuous) will have to be continuous: should it jump discontinuously, the same will have to be true of $\dot{n}_{S1}/n_{S1}$ and $\dot{n}_2/n_2$ individually, which implies a discontinuous jump in L_r (since $\dot{n}_{M1}/n_{M1}$ equals 0, and using equation (A2) can be seen to be continuous, at the transition point), violating the labour market equilibrium condition (36) (all other terms in (36) can be shown to be continuous). Thus, $\dot{n}_{S1}/n_{S1}$ and $\dot{n}_2/n_2$ will each have to be continuous as well, and so will $\dot{Z}/Z$, implying that the slope of the saddle-path at the transition point is also continuous. The same applies at any other transition point between two regimes.

Formally, then, a backward transition from Regime 1 to 2 will occur if the saddle-path cuts the $\dot{n}_{M1}/n_{M1}=0$ locus of Regime 2 at a point at which $\dot{n}_{S1}/n_{S1}$ from either regime (which are equivalent at a point at which $\dot{n}_{M1}/n_{M1}=0$) is positive. It is easily shown that this requires that the saddle-path cut the $\dot{n}_{M1}/n_{M1}=0$ locus anywhere to the right of a ray from the origin given by

(A21) $Z \;=\; [\theta(1-\alpha_{S1})][\beta_1(1-\alpha_{M1})]^{-1}G$

From equations (A2) and (61) it may be seen that the horizontal intercept of the $\dot{n}_{M1}/n_{M1}=0$ locus of Regime 2 lies strictly between the origin and the horizontal intercept of the $\dot{G}/G=0$ locus of Regime 1 (the saddle-path in Regime 1 must lie to the right of this latter locus). Denoting the intersection point between this latter locus and the above ray by point H, it follows that the $\dot{n}_{M1}/n_{M1}=0$ locus must pass to the right of point H, failing which the saddle-path will cut it to the *left* of the above ray. It turns out that in the negatively-sloped case, which requires $\beta_1(1-\alpha_{M1})<1/3$, this cannot be satisfied, while in the other cases, depending on parameter values, it can. After straightforward manipulations we obtain the following necessary lower bound on $\beta_1(1-\alpha_{M1})$ if a transition from 1 to 2 is to occur:

(A22) $\beta_1(1-\alpha_{M1}) \;\geq\; max.\,[1/3,\ 1/\{3+(a\rho)^{-1}-[\theta(1-\alpha_{S1})]^{-1}\}]$

The second argument on the right exceeds 1/3 when $\theta(1-\alpha_{S1})$ is at its lower bound given by the right side of (64), falls as $\theta(1-\alpha_{S1})$ rises, and may fall below 1/3 when $\theta(1-\alpha_{S1})$ reaches a hypothetical maximum of 1. In the region where the second argument does not fall below 1/3, $\beta_1(1-\alpha_{M1})$ and $\theta(1-\alpha_{S1})$ are thus 'substitutes'. A high $\beta_1(1-\alpha_{M1})$ implies that innovation in *M*1 will persist longer in the growth process, and a high $\theta(1-\alpha_{S1})$ that innovation in $S1$ will commence earlier, and either of these will conduce to these innovation phases overlapping with each other.

We next show that, proceeding backwards into Regime 2 from the transition point, the saddle-path will remain negatively-sloped – it cannot turn rightwards, implying that, proceeding forwards now, both G and Z are falling, nor can it turn downwards, implying that G and Z are rising forwards. In the former case, a vertical line drawn at some value of G to the right of the hypothetical turning-point would intersect both arms of the saddle-path, once when $\dot{G}/G$ is negative, and again, at a *lower* value of Z, when $\dot{G}/G$ is positive: however, from (A4) $\dot{G}/G$ is an increasing function of Z, so such a scenario is not possible. A precisely analogous argument can be employed to exclude the latter case should the coefficient, of G now, in (A5) be positive or zero. If the coefficient is negative, that of Z can be seen to be negative also, and the $\dot{Z}/Z = 0$ locus in this case will be negatively-sloped, and will have a negative horizontal intercept. Setting up the resulting phase-diagram, it is easily shown that if the saddle-path is 'initially' negatively-sloped (as it is at the transition point from Regime 1 to 2), it will going backwards always remain so, and thus not turn down.

Proceeding backwards along the saddle-path in Regime 2, a point will be reached where $\dot{n}_{S1}/n_{S1}$ = 0, since from (A1) and (A4) the vertical intercept of the $\dot{n}_{S1}/n_{S1} = 0$ locus lies below that of the $\dot{G}/G = 0$ locus. From (A1) and (A3), if $[\theta_2(1-\alpha_2) - \theta(1-\alpha_{S1})]G + \beta_2(1-\alpha_2)Z$ is positive at that point, $\dot{n}_2/n_2$ will be positive there, and conversely. Although we have earlier argued that $\theta_2(1-\alpha_2) - \theta(1-\alpha_{S1})$ is likely to be negative, we also argued that β_2 exceeds ω and, *a fortiori*, θ_2, and moreover proceeding up the saddle-path G is declining and Z is rising. Thus, it appears plausible to suppose that $\dot{n}_2/n_2$ remains positive, and so the backwards crossover is from Regime 2 to 3: the alternative case is quite straightforward, with the backwards trajectory transiting from Regime 2 to Regime 4, whence from the resulting phase diagram it has to continue backwards in the same northwest direction, and then, upon crossing the $\dot{n}_{S1}/n_{S1}$ = 0 locus, transition to Regime 6 (analyzed below) and remain there, all the way towards the vertical axis if necessary.

Again, it can be shown that in Regime 3 the backwards trajectory will maintain a northwest movement.[1] It will then, as it converges towards the vertical axis, either remain all the way in Regime 3, or transition to Regime 6, in which $\dot{n}_2/n_2$ is 0 and only $\dot{n}_{M1}/n_{M1}$ is positive, and remain there.[2]

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[1] From (A9) it is not possible to have a parameter configuration such that the coefficient of *G/a* is negative and that of *Z/a* positive. In all other cases, with the backwards crossover from Regime 2 to 3 occuring in a northwest direction, the resulting phase diagrams show that the trajectory cannot transition to any other phase.

[2] In Regime 6, all three loci – $\dot{G}/G = 0$, $\dot{n}_{M1}/n_{M1} = 0$, and $\dot{Z}/Z = 0$ – will intersect at a common point in the positive quadrant, and will all be negatively-sloped, with the flattest (the algebraically largest) slope belonging to the $\dot{G}/G = 0$ locus, followed by the $\dot{Z}/Z = 0$ and then the $\dot{n}_{M1}/n_{M1} = 0$ loci. The backward saddle-path trajectory will thus extend all the way towards the vertical axis. It should be mentioned that there appears to exist a slight

Lastly, we briefly consider the case in which the first backward transition is from Regime 1 to 5 – with $\dot{n}_{S1}/n_{S1}$ falling to 0 before $\dot{n}_{M1}/n_{M1}$ becomes positive, if ever, so that only $\dot{n}_2/n_2$ is positive. It is readily shown that in Regime 5, all three loci – $\dot{G}/G=0$, $\dot{Z}/Z=0$, and $\dot{n}_2/n_2=0$ – are negatively sloped, and do not intersect in the positive quadrant, with the $\dot{n}_2/n_2=0$ locus being outermost, followed by the $\dot{Z}/Z=0$ locus. Between the $\dot{G}/G=0$ and $\dot{Z}/Z=0$ loci, the backwards trajectory will have a northwest movement, and cannot transition to any other region in this regime's phase diagram. Proceeding backwards, the trajectory may either remain in Regime 5 all the way towards the vertical axis, or transition to Regime 3, in which $\dot{n}_{M1}/n_{M1}$ also becomes positive. A necessary condition for the latter is $\beta_1(1-\alpha_{M1}) > \rho[2\rho+(1/a)]^{-1}$, which ensures that the vertical intercept of the $\dot{n}_{M1}/n_{M1}=0$ locus of Regime 3 is above that of the $\dot{G}/G=0$ locus of Regime 5, noting that the two regimes are equivalent at any point along the $\dot{n}_{M1}/n_{M1}=0$ locus. (At the same time, $\beta_1(1-\alpha_{M1})$ should not be so high that the first backwards transition was from Regime 1 to 2, in which case the preceding analysis applies.) Should the system cross into Regime 3, the preceding analysis then applies, from then backwards. This completes the proof of the Proposition. ‖

B. First-Order Conditions of the Social Planner's Optimization Problem, and Some Preliminary Relationships

(A23) $\dfrac{\partial H_S}{\partial L_M} = (1-\beta_1-\beta_2)Y_M^{1-\sigma}/L_M - (q_2a^{-1}n_2+\phi) = 0$, where

(A24) $Y_M = A_M L_M^{1-\beta_1-\beta_2} n_{M1}^{\beta_1/\alpha_{M1}} x_{M1}^{\beta_1} n_2^{\beta_2/\alpha_2} x_{M2}^{\beta_2}$

(A25) $\dfrac{\partial H_S}{\partial L_S} = \tau(1-\theta-\theta_2)/L_S - (q_2a^{-1}n_2+\phi) = 0$

(A26) $\dfrac{\partial H_S}{\partial x_{M1}} = \beta_1 Y_M^{1-\sigma}/x_{M1} - (q_2a^{-1}n_2+\phi)n_{M1} = 0$

(A27) $\dfrac{\partial H_S}{\partial x_{M2}} = \beta_2 Y_M^{1-\sigma}/x_{M2} - (q_2a^{-1}n_2+\phi)n_2 = 0$

(A28) $\dfrac{\partial H_S}{\partial x_{S1}} = \tau\theta/x_{S1} - (q_2a^{-1}n_2+\phi)n_{S1} = 0$

(A29) $\dfrac{\partial H_S}{\partial x_{S2}} = \tau\theta_2/x_{S2} - (q_2a^{-1}n_2+\phi)n_2 = 0$

(A30) $\dfrac{\partial H_S}{\partial L_{rS1}} = q_{S1}a^{-1}n_{S1} - (q_2a^{-1}n_2+\phi) \le 0$; $= 0$ if $L_{rS1} > 0$

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theoretical possibility that, instead of transiting backwards from Regime 3 to 6, the trajectory transits from Regime 3 to 5, in which $\dot{n}_{M1}/n_{M1}=0$ and only $\dot{n}_2/n_2 > 0$. A sufficient condition to exclude this is $\beta_1(1-\alpha_{M1}) > 1/2$. The likelihood of this slight theoretical possibility is somewhat greater if $\beta_2(1-\alpha_2)$ is close to or exceeds $\beta_1(1-\alpha_{M1})$, but this is itself unlikely since it would tend to imply a low share of labour in Y_M. We thus ignore this possibility, also because it does not affect the Stages of Growth pattern identified in the text.

(A31) $\dfrac{\partial H_S}{\partial L_{rM1}} = q_{M1}a^{-1}n_{M1} - (q_2 a^{-1}n_2 + \phi) \leq 0\,;\ = 0 \text{ if } L_{rM1} > 0$

(A32) $\dot{q}_{M1} = \rho q_{M1} - \dfrac{\partial H_S}{\partial n_{M1}} = \rho q_{M1} - [\beta_1 Y_M^{1-\sigma} / \alpha_{M1}n_{M1} + q_{M1}a^{-1}L_{rM1} - (q_2 a^{-1}n_2 + \phi)x_{M1}]$

(A33) $\dot{q}_{S1} = \rho q_{S1} - \dfrac{\partial H_S}{\partial n_{S1}} = \rho q_{S1} - [\,\tau\theta\alpha_{S1}^{-1}n_{S1}^{-1} + q_{S1}a^{-1}L_{rS1} - (q_2 a^{-1}n_2 + \phi)x_{S1}]$

(A34) $\dot{q}_2 = \rho q_2 - \dfrac{\partial H_S}{\partial n_2} = \rho q_2 - [(\beta_2 Y_M^{1-\sigma} + \tau\theta_2)/\alpha_2 n_2 + q_2 L_{r2}/a -$

$(q_2 a^{-1}n_2 + \phi)(x_{M2} + x_{S2})]$

(A35) $\phi \geq 0;\ \ \phi L_{r2} = 0$

(A36) $Lim_{t\to\infty}e^{-\rho t}q_{S1}n_{S1} = Lim_{t\to\infty}e^{-\rho t}q_{M1}n_{M1} = Lim_{t\to\infty}e^{-\rho t}q_2 n_2 = 0\,,$

as well as the equations of motion (23), with L_{r2} replaced using the labour-market-clearing condition.

The foregoing conditions give rise to some useful preliminary 'regime-independent' relationships (not dependent on whether any given $\dot{n}_k / n_k$ is positive or 0):

(A37) $(1-\beta_1-\beta_2)Y_M^{1-\sigma}/L_M = (q_2 a^{-1}n_2 + \phi) = \beta_1 Y_M^{1-\sigma}/n_{M1}x_{M1} = \beta_2 Y_M^{1-\sigma}/n_2 x_{M2} =$
$\tau(1-\theta-\theta_2)/L_S = \tau\theta/n_{S1}x_{S1} = \tau\theta_2/n_2 x_{S2}$

(A38) $Y_M = (q_2 a^{-1}n_2 + \phi)^{-1/\sigma}\, n_{M1}^{\beta_1(1-\alpha_{M1})/\sigma\alpha_{M1}} n_2^{\beta_2(1-\alpha_2)/\sigma\alpha_2}$

(obtained from equations (1)-(3), (A37), and the same normalization of A_M (= $(1-\beta_1-\beta_2)^{-(1-\beta_1-\beta_2)}\beta_1^{-\beta_1}\beta_2^{-\beta_2}$) as employed previously),

(A39) $\dot{q}_{S1}/q_{S1} + \dot{n}_{S1}/n_{S1} = \rho - \tau\theta(q_{S1}n_{S1})^{-1}(\alpha_{S1}^{-1} - 1)$

(A40) $\dot{q}_2/q_2 + \dot{n}_2/n_2 = \rho - (q_2 n_2)^{-1}(\alpha_2^{-1} - 1)[\beta_2 R + \tau\theta_2]$, where

(A41) $R = Y_M^{1-\sigma}$

(A42) $\dot{q}_{M1}/q_{M1} + \dot{n}_{M1}/n_{M1} = \rho - (q_{M1}n_{M1})^{-1}(\alpha_{M1}^{-1} - 1)\beta_1 R$

Finally, (A39) is a nonlinear first-order differential equation in $q_{S1}n_{S1}$ alone, which has either the degenerate solution

(A43) $q_{S1}n_{S1} = \tau\theta(\alpha_{S1}^{-1} - 1)/\rho$

for all t, or in the non-degenerate case can be solved to yield

(A44) $q_{S1}n_{S1} = [\tau\theta(\alpha_{S1}^{-1} - 1)/\rho] + e^{\rho(t+C)}/\rho\,,$

where C is an arbitrary constant. The latter solution does not satisfy the transversality condition, however, and hence (A43) holds – irrespective of regime.

C. Implications of the Foregoing Conditions

(a) Equality of $q_{S1}n_{S1}$ and $q_2 n_2$ over a strictly positive time interval implies, as pointed out in the article, that their respective rates of change are equal over this interval, which from (A39) and (A40) implies that R (= $Y_M^{-(\sigma-1)}$) is *fixed* in this interval, and given by (73) in the article. From (A38), fixity of R, and hence Y_M, and of $q_2 n_2$ (= $q_{S1}n_{S1}$, fixed by (A43)) implies that n_2 has also to be fixed, as asserted in the text.

(b) Analogously, from (A30)-(A31) and (A39)-(A42) it is easily seen that if any two of $\dot{n}_{S1}/n_{S1}$, $\dot{n}_2/n_2$, and $\dot{n}_{M1}/n_{M1}$ are strictly positive over a strictly positive time interval (requiring that the rates of change of the corresponding $q_k n_k$ be equal) then R, and hence Y_M, cannot change during that interval.

D. Further Analysis of the Socially Optimal Path

We first study Regime SP5, in which only $\dot{n}_2/n_2$ can be positive. Letting W_2 and $N(n_{M1},n_2)$ denote $(q_2 n_2)^{-1}$ and $n_{M1}^{-\beta_1(1-\alpha_{M1})/\alpha_{M1}} n_2^{-\beta_2(1-\alpha_2)/\alpha_2}$ respectively, and using (A40) and other equations, Regime SP5 is characterized by:

(A45) $\dot{W}_2/W_2 \;=\; (\alpha_2^{-1}-1)\{\beta_2 a^{1/\sigma-1} W_2^{1/\sigma} N(\hat{n}_{M1},n_2)^{(1-\sigma^{-1})} + \tau\theta_2 W_2\} - \rho$

(A46) $\dot{n}_2/n_2 \;=\; a^{-1} - \tau W_2 - a^{1/\sigma-1} W_2^{1/\sigma} N(\hat{n}_{M1},n_2)^{(1-\sigma^{-1})}$

and a further equation, in $\dot{W}_{M1}/W_{M1}$ ((A47) below, with n_{M1} fixed and n_2 varying: it should be noted that the $\dot{W}_i/W_i$ differential equations (i = $S1$, 2, $M1$) are all regime-independent, but which n_i variables in those equations are fixed and which changing are regime-dependent). With n_{M1} fixed at $\hat{n}_{M1}$ (to be determined subsequently), (A45) and (A46) generate a phase-diagram in n_2-W_2 space. Both $\dot{W}_2/W_2 = 0$ and $\dot{n}_2/n_2 = 0$ loci are positively sloped, and we obtain Fig 2 below.

Since, as will be seen, SP5 will going forward be succeeded by SP1, the steady-state value of W_2, W_2^*, is simply the inverse of $q_{S1}n_{S1}$ as given in (A43). It is easily shown that strict positiveness of the steady-state value of L_r (*fn.* 25 of main text) implies that the two loci in Fig 2 intersect in the positive quadrant, at $\hat{W}_2$ say, and that $W_2^* < \hat{W}_2$. Thus, in SP5 there exists a unique north-east trajectory commencing at n_{20} (not drawn) and leading to point A in Fig 2, and when A is attained there will be an instantaneous switch to SP1, entailing no change in W_2 or W_{S1} (= $(q_{S1}n_{S1})^{-1}$), but an instantaneous switch in labour devoted to *R&D* from innovation in set 2 to set $S1$. (A rising W_2 during SP5 implies a falling q_2n_2, until it falls to equality with the unchanging $q_{S1}n_{S1}$, at which point the switch occurs: note also that the steady-state point A is *not* at the intersection of the two loci, unlike in customary analyses.) The analysis is of course conditional on the value of $\hat{n}_{M1}$: a higher value of this would cause the loci to shift leftwards, except at the origin, but the final value of W_2 remains at W_2^*, while the final value of n_2 will be lower.

We next examine Regime SP6, in which only $\dot{n}_{M1}/n_{M1} \geq 0$. Using (A42) and other equations, and letting W_{M1} denote $(q_{M1}n_{M1})^{-1}$, we now have

(A47) $\dot{W}_{M1}/W_{M1} \;=\; \beta_1(\alpha_{M1}^{-1}-1) a^{1/\sigma-1} W_{M1}^{1/\sigma} N(n_{M1},n_{20})^{(1-\sigma^{-1})} - \rho$

(A48) $\dot{n}_{M1}/n_{M1} \;=\; a^{-1} - \tau W_{M1} - a^{1/\sigma-1} W_{M1}^{1/\sigma} N(n_{M1},n_{20})^{(1-\sigma^{-1})}$

and (A45) above, with n_2 now fixed and n_{M1} varying. With n_2 fixed in this regime at n_{20} we have a phase-diagram in n_{M1}–W_{M1} space. We require that the productivity term for set $M1$ $\beta_1(\alpha_{M1}^{-1}-1)$ be large enough that $\rho/\beta_1(\alpha_{M1}^{-1}-1) < a^{-1}$, failing which this regime cannot form part of the optimal solution[3]: the resulting phase-diagram is very similar to Fig 2, with n_2 and W_2 replaced by n_{M1} and W_{M1} and with the $\dot{n}_{M1}/n_{M1}=0$ locus cutting the $\dot{W}_{M1}/W_{M1}=0$ locus from above (but with the latter locus rising continually, without a finite asymptote). We also note that if, proceeding forwards now, SP5 is to succeed SP6, this will have to occur at a point at which the rising W_2 path cuts the rising W_{M1} from above, after which W_{M1} rises above W_2 ($q_{M1}n_{M1}$ falls below $q_2 n_2$), and both converge to their respective steady-state values at exactly the same finite time, the time at which the switch to SP1 occurs.[4]

Heuristically, it is helpful to visualize these developments by using horizontal and vertical *cross-sections* of a three-dimensional phase-diagram (Fig 3 below). In analyzing this, we note first that our system is 'sequentially block-recursive': in Regime SP5 (A45) and (A46) form a self-contained dynamic system in n_2–W_2 space, while the behaviour of W_{M1} depends on that of n_2 and W_2; in SP6 (A47) and (A48) are self-contained, and influence the behaviour of W_2. In the horizontal n_2–W_2 plane in Fig 3, drawn for SP5, we have, to avoid clutter, drawn just the $\dot{W}_2/W_2=0$ locus, as well as the trajectory *BA* commencing from n_{20} and ending at the steady-state point *A*. In the vertical W_2–W_{M1} plane we have for the same Regime drawn a cross-section of the contemporaneous trajectory *CD*, which has to reach the steady-state point *D* at precisely the same time as point *A* below is attained (the $\dot{W}_{M1}/W_{M1}=0$ locus for SP5 is not drawn). Notice that point *D* lies above the 45° line, since $W_{M1}^* > W_2^*$. We now come to the critical point. Proceeding backwards from the steady-state points *A* and *D* along the respective cross-sections to the start point of Regime SP5, which is where $n_2 = n_{20}$,[5] what assurance is there that, at the start point, point *C* will lie on the 45° line, as it must (as explained earlier, the crossover into SP5 occurs exactly when $W_{M1}=W_2$)? To assure this, $\hat{n}_{M1}$ has to be chosen accordingly, which serves to pin down the solution for this variable, denoted n_{M1}^*. Thus, the preceding Regime SP6 will have to prevail until n_{M1} rises to this value, and the final steady-state value of n_2 is then determined by this and the steady-state values of Y_M and W_2. It follows that n_{M1}^* is a function of the initial condition n_{20}: the steady-state values of n_{M1} and n_2 are thus not independent of this particular initial condition of the model! We also note (a) that as $\hat{n}_{M1}$ is varied the paths *BA* and *CD* will change, except for their ending values W_2^* and W_{M1}^* and the requirement that point *B* lie somewhere along the $n_2 = n_{20}$ line, so as to achieve equality of the start values of W_2 and W_{M1} of SP5 (when $\hat{n}_{M1} = n_{M1}^*$), and (b) at the very beginning of the optimal growth process, when SP6 commences, the values of the co-state variables q_2 and q_{M1} will as usual have to be

[3] n_{M1} could then only rise if W_{M1} falls, and W_{M1} would then continue to fall even after n_{M1} ceases rising, at an rate that eventually violates the transversality condition.

[4] The steady-state value W_{M1}^* $(=\rho/\beta_1(\alpha_{M1}^{-1}-1)R^*$, at which $\dot{W}_{M1}/W_{M1}=0)$ must thus exceed W_2^*, which can be shown to require that $\beta_2(\alpha_2^{-1}-1)$ not be too far below $\beta_1(\alpha_{M1}^{-1}-1)$ (otherwise the system would transition directly from SP6 to SP1, and the final value of *R* would be different): it must however be below $\beta_1(\alpha_{M1}^{-1}-1)$, otherwise SP6 would not be part of the optimal path, which would simply transition from SP5 to SP1.

[5] Since n_2 is constant during SP6, which as pointed out below is the first phase of the optimal growth process, its value n_{20} at the start of SP5 is the same as its value at the commencement of the entire optimal growth process.

chosen to ensure that the resulting W_2 and W_{M1} paths arrive at the above-indicated SP5 start value at the same time.

Formal validation of the foregoing heuristic argument turns out to be a highly complicated affair, and in Section D below we linearize the model in the SP5 phase and provide a precise algebraic solution for n^*_{M1}, clearly showing its dependence on n_{20}. Finally, using (A45) and (A47), at any switch-point between SP5 and SP6 (at which $W_{M1} = W_2$), we have $\dot{W}_{M1}/W_{M1} - \dot{W}_2/W_2 = \{[\beta_1(\alpha_{M1}^{-1}-1) - \beta_2(\alpha_2^{-1}-1)]R - \tau\theta_2(\alpha_2^{-1}-1)\}\, W_2$. The square-bracketed term is positive (*fn.* 4 above), and R is declining as Y_M grows. Thus, it is possible for the entire term in braces to be positive at a low value of Y_M, implying a switch from set M1 to 2, and to be negative at another hypothetical switch point at a higher value of Y_M, but it is not possible for there to then be a later switch to set 2 again. Given $W^*_{M1} > W^*_2$, however, and continuity of the optimal W_i trajectories, innovation in set 2, and not set M1, must occur immediately prior to the final switch to SP1. Thus as asserted the forward sequence of stages is indeed from SP6 to SP5 to SP1.

E. Linearization of Regime SP5 of the Social Planner's Problem

Terminal values W^*_{M1} and W^*_2 of Regime SP5 have been provided earlier, and n^*_2, and thence n^*_{M1}, are to be determined. Letting the superscript '*d*' denote actual minus steady-state value, and x the column vector $(W^d_{M1}, W^d_2, n^d_2)'$ (primes denote transpose), linearization of (A47) (with $N(n_{M1}, n_{20})^{(1-\sigma^{-1})}$ replaced by $N(\hat{n}_{M1}, n_2)^{(1-\sigma^{-1})}$), (A45) and (A46) around the steady state yields, in matrix notation:

(A49) $\dot{x} = Ax + b$,

where the elements of the matrix A are

(A50) $a_{11} = W^*_{M1}\beta_1(\alpha_{M1}^{-1}-1)R^* \quad (= \rho)$

(A51) $a_{12} = -\rho W^*_{M1}(1-\sigma^{-1})/W^*_2$

(A52) $a_{13} = -\rho W^*_{M1}\beta_2(\alpha_2^{-1}-1)(1-\sigma^{-1})/n^*_2$

(A53) $a_{21} = 0$

(A54) $a_{22} = (\alpha_2^{-1}-1)[\beta_2\sigma^{-1}R^* + \tau\theta_2]W^*_2$

(A55) $a_{23} = -[\beta_2(\alpha_2^{-1}-1)W^*_2]^2(1-\sigma^{-1})R^*/n^*_2$

(A56) $a_{31} = 0$

(A57) $a_{32} = -n^*_2[\tau + \sigma^{-1}R^*]$

(A58) $a_{33} = a^{-1}L^*_r + (1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W^*_2 R^*$, where

(A59) $L^*_r = 1 - aW^*_2[\tau + R^*]$,

and b is the column vector $(0, 0, n^*_2 L^*_r)'$. From the third row of (A49) we thus note that $\dot{n}^d_2$ is not 0 at the steady-state value of x $(= (0,0,0)')$, but $\dot{n}^d_2$ here refers to the left-hand derivative of n^d_2: as explained earlier, there occurs an instantaneous switch to SP1 at the steady-state point, so that the right-hand derivative of n^d_2 is indeed 0.

(A49) is a block-recursive system, with $\dot{W}^d_2$ and $\dot{n}^d_2$ not dependent on W^d_{M1}: as such, one eigenvalue of the system is simply a_{11} $(= \rho)$, and the other two are the eigenvalues of the lower right 2 x 2 sub-matrix of A. It considerably simplifies the algebra if we now suppose that the determinant of this sub-matrix, $a_{22}a_{33} - a_{32}a_{23}$, is 0, so that these two eigenvalues are equal: given that our aim is in part to simply illustrate the existence of Initial Condition Dependence in the model, and that a large number of parameters appears in this determinant, so that setting it to 0 does not appear to entail particularly severe economic restrictions, such an algebraic simplification appears acceptable. With repeated roots the solution of the (W_2^d, n_2^d) block takes the form (for convenience, the start time of Regime SP5 is set at 0):

$$\text{(A60)} \quad W_2^d \;=\; Le^{(a_{22}+a_{33})t} - \frac{a_{23}b_3}{a_{22}+a_{33}}t + X\,,$$

$$\text{(A61)} \quad n_2^d \;=\; \frac{a_{32}}{a_{22}}Le^{(a_{22}+a_{33})t} + \frac{a_{22}b_3}{a_{22}+a_{33}}t - [\frac{b_3}{a_{22}+a_{33}} + \frac{a_{32}}{a_{33}}X\,],$$

where L and X are to be determined. We may then solve for W_{M1}^d:

$$\text{(A62)} \quad W_{M1}^d \;=\; N_1 e^{a_{11}t} + \frac{(a_{12} + a_{13}a_{32}a_{22}^{-1})L}{L_r^*/a}e^{(a_{22}+a_{33})t} + \frac{b_3(a_{12}a_{23} - a_{13}a_{22})}{a_{11}(a_{22}+a_{33})}t - \frac{(a_{12}a_{33} - a_{13}a_{32})}{a_{11}a_{33}}X\,,$$

with N_1 to be determined. Our system thus has 5 unknowns – n_2^*, L, X, N_1, *and* $\hat{t}$, say, the terminal time, at which the switch from SP5 to SP1 occurs. We also have 5 equations, namely each of (W_{M1}^d, W_2^d, n_2^d) has to equal 0 at $t = \hat{t}$, W_{M1} (*not* W_{M1}^d) has to equal W_2 at $t = 0$, and $n_2^* + n_2^d(0))$ has to equal the given initial value n_{20}. Thus, n_2^* depends on n_{20}, as we will now explicitly show.

After some exceedingly tedious derivations (earlier as well), we finally obtain the following solutions. L, which is negative, is, conditional on n_2^*, the solution of an implicit equation:

$$\text{(A63)} \quad (-L)e^{\frac{(-L)(a^{-1}L_r^*+\rho)^2[\tau+\sigma^{-1}R^*]}{a^{-1}L_r^*[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]^2} + \frac{(a^{-1}L_r^*)^{-1}(a^{-1}L_r^*+\rho)^2 z-[a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]}{[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]}} = \frac{a^{-1}L_r^*[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*][a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]}{(a^{-1}L_r^*+\rho)^2[\tau+\sigma^{-1}R^*]},$$

where $z = 1-(n_{20}/n_2^*)$: indeed, n_2^* will appear in this form in all our solution equations. We also have:

$$\text{(A64)} \quad \hat{t} \;=\; \frac{(-L)(a^{-1}L_r^*+\rho)[\tau+\sigma^{-1}R^*]}{a^{-1}L_r^*[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]^2} + \frac{(a^{-1}L_r^*+\rho)z}{(a^{-1}L_r^*)[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]} - \frac{[a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]}{[\rho-(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*](a^{-1}L_r^*+\rho)}$$

$$\text{(A65)} \quad X \;=\; \frac{[a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]L}{(\alpha_2^{-1}-1)[\beta_2\sigma^{-1}R^*+\tau\theta_2]W_2^*} + \frac{[a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]a^{-1}L_r^*}{[\tau+\sigma^{-1}R^*](a^{-1}L_r^*+\rho)} - \frac{[a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*]}{[\tau+\sigma^{-1}R^*]}z$$

$$\text{(A66)} \quad N_1 \;=\; \rho W_{M1}^*(1-\sigma^{-1})(\alpha_2^{-1}-1)\,\{\frac{\tau[\beta_2-\theta_2](-L)}{(\alpha_2^{-1}-1)[\beta_2\sigma^{-1}R^*+\tau\theta_2]W_2^*a^{-1}L_r^*}e^{a^{-1}L_r^*\hat{t}} - \beta_2(a^{-1}L_r^*+\rho)^{-2}a^{-1}L_r^*e^{-\rho\hat{t}}\}$$

(A67)

$$\{\frac{\beta_2(\alpha_2^{-1}-1)W_2^*\tau\theta_2}{[\beta_2\sigma^{-1}R^*+\tau\theta_2]}+(\frac{W_{M1}^*}{W_2^*}-1)[\beta_2(\alpha_2^{-1}-1)W_2^*+\frac{a^{-1}L_r^*+(1-\sigma^{-1})\beta_2(\alpha_2^{-1}-1)W_2^*R^*}{[\tau+\sigma^{-1}R^*]}]\}.$$

$$(1-\sigma^{-1})[\frac{a^{-1}L_r^*}{a^{-1}L_r^*+\rho}-z] \ + \ W_2^*(\frac{W_{M1}^*}{W_2^*}-1) \ = \ (-L)\frac{a^{-1}L_r^*+\rho}{(\alpha_2^{-1}-1)[\beta_2\sigma^{-1}R^*+\tau\theta_2]W_2^*}.$$

$$\{\frac{W_{M1}^*}{W_2^*}\frac{\tau\theta_2}{\beta_2R^*}+(\frac{W_{M1}^*}{W_2^*}-1)\}$$

(A63) and (A67) jointly determine (–*L*) and *z*, after which the remaining unknowns are determined. We thus study these two equations next. Inspection of (A63) shows that if $z = 0$, (–*L*) has an explicit solution that is simply given by the right side of (A63) (the term in *e* equals 1 at this value of (–*L*)): the solution is unique, since the left side is strictly increasing in (–*L*) while the right is constant. Successive differentiation of (A63) with respect to *z* also shows that (–*L*) is a decreasing convex function of *z*, and the same is thus true of the right side of (A67), denoted *RHS*. The left-side of (A67), *LHS*, is a decreasing linear function of *z*, and with some algebraic effort it can be shown that the vertical intercept (at $z = 0$) of *LHS* strictly exceeds that of *RHS*, given that $W_{M1}^* > W_2^*$: in the hypothetical case that these are equal, RHS will be tangent to LHS at $z = 0$, and for $z > 0$ will lie above it. For $W_{M1}^* > W_2^*$ there is thus a unique solution for *z*, denoted z^* (<1[6]).

To conclude, z^* and n_{20} will jointly determine n_2^*, and n_2^*, R^*, and W_2^* will jointly determine n_{M1}^*. We assume that the initial value of n_{M1} is less than n_{M1}^*, so that the first phase of the optimal path is indeed SP6: failing this, the system will simply transit from SP5 to SP1, and the final value of n_2 will be determined residually from R^*, W_2^*, and the initial n_{M1}.

F. Policy for the Decentralized Economy

Let $(1 - h_k)$ ($k = S1, M1, 2$) denote the subsidies to purchasers of inputs from group *k*, who thus now pay h_kp_k for the respective composite inputs, and $(1 - b_k)$ the subsidies to employers of labour in *R&D* activities, who thus face wage costs of b_kw. These should be inserted in the relevant equations earlier, and in Regime 1 we finally arrive at

(A68) $\dot{n}_{S1}/n_{S1} \ = \ (1/2a)\{1-[1-\theta(1-\alpha_{S1})/h_{S1}b_{S1}+\theta_2(1-\alpha_2)/h_2b_2]G \ -$
$[1+\beta_2(1-\alpha_2)/h_2b_2]Z\}$, where

(A69) $Z \ = \ \tau^{-1/\sigma}w^{-1/\sigma}(\alpha_{M1}/h_{M1})^{-\beta_1(1-1/\sigma)}n_{M1}^{-\beta_1(1-1/\sigma)(1-\alpha_{M1})/\alpha_{M1}}.$
$(\alpha_2/h_2)^{-\beta_2(1-1/\sigma)}n_2^{-\beta_2(1-1/\sigma)(1-\alpha_2)/\alpha_2}$,

(A70) $\dot{n}_2/n_2 \ = \ (1/2a)\{1-[1-\theta(1-\alpha_{S1}h_{S1}^{-1})+\theta(1-\alpha_{S1})/h_{S1}b_{S1}-\theta_2(1-\alpha_2h_2^{-1})-$
$\theta_2(1-\alpha_2)/h_2b_2)]G-[1-\beta_1(1-\alpha_{M1}/h_{M1})-\beta_2(1-\alpha_2/h_2)-\beta_2(1-\alpha_2)/h_2b_2)]Z\}$

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[6] As W_2^* falls below W_{M1}^* z^* will gradually increase, to some value less than unity: $z^* = 1$ implies n_2^* is infinite, which is not possible since $\hat{t}$ would still be finite then.

To attain the social optimum, it is necessary for the values of $\dot{n}_k / n_k$, $k = S1, 2,$ from these equations to coincide with the corresponding values in Regime SP1[7] ($\dot{n}_2 / n_2$ is of course 0 in SP1). A first requirement for this is that $h_k = \alpha_k$ – a standard result, which applies also in other regimes. Also, from inspection of the relevant equations one might conjecture that $\tau b_{S1}^{-1} v_{S1} n_{S1}$ $(= \tau b_2^{-1} v_2 n_2)$ should correspond to $q_2 n_2$ (we now have that $w = (ab_{S1})^{-1} v_{S1} n_{S1} =$ $(ab_2)^{-1} v_2 n_2$ in Regime 1, which justifies the preceding bracketed equality): we later verify that this conjecture is correct. Next, we logarithmically differentiate the equation $\tau b_{S1}^{-1} v_{S1} n_{S1} =$ $q_2 n_2$ with respect to t (noting that in SP1 $q_2 n_2 = q_{S1} n_{S1}$ is constant), use (A68) and its required equality with the socially optimal $\dot{n}_{S1} / n_{S1}$, and also note that in (55) the second right-hand term should be divided by h_{S1}: we then end up with a differential equation in b_{S1} alone, whose only stable solution is a constant one,

(A71) $$b_{S1}^* = \rho / \{\rho + a^{-1} - \rho(1 - \theta_2 \beta_2^{-1})[\theta(\alpha_{S1}^{-1} - 1)]^{-1} - \rho[\beta_2(\alpha_2^{-1} - 1)]^{-1}\}$$

The terms in braces above, excluding the first term ρ, add up to L_r / a which is positive (*fn.* 25 of main text), so that $b_{S1}^* \in (0,1)$: the optimal subsidy to *R&D* in *S*1 is strictly positive. Substituting b_{S1}^* into (A70) and equating private and socially optimal $\dot{n}_2 / n_2$'s, we derive the optimal b_2, b_2^*, which is 1 – no subsidy should be given to *R&D* in set 2, which is of course consistent with the zero value of $\dot{n}_2 / n_2$ in this Regime.

Next, we consider the decentralized implementation of SP5, through policy intervention in Regime 5. Since only $\dot{n}_2 / n_2 \geq 0$ in these regimes, only the equalities $w = (ab_2)^{-1} v_2 n_2 = (a \tau W_2)^{-1}$ must hold, and differentiating the latter equality (note that this equality implies differentiability of b_2 over time) with respect to t and substituting from earlier equations we arrive at

(A72) $$\dot{b}_2 / b_2 = a^{-1} - \tau W_2 - a^{1/\sigma - 1} W_2^{1/\sigma} N(\hat{n}_{M1}, n_2)^{(1-\sigma^{-1})} - (\alpha_2^{-1} - 1)[\tau \theta_2 W_2 + \beta_2 a^{1/\sigma - 1} W_2^{1/\sigma} N(\hat{n}_{M1}, n_2)^{(1-\sigma^{-1})}](b_2^{-1} - 1),$$

together with Regime SP5's differential equations in W_2 and n_2. Since b_2 is continuous, it must converge to 1 by the end of SP5: also the first three expressions on the right of (A72) are jointly positive, since they equal L_r/a. If σ is close to 1, so that the movement of $W_2^{1/\sigma} N(\hat{n}_{M1}, n_2)^{(1-\sigma^{-1})}$ is governed by the rising movement of W_2, $\dot{b}_2 / b_2$ will have to be positive throughout (which requires from (A72) that b_2 cannot be below 1 by too much to begin with), since otherwise $\dot{b}_2 / b_2$ will become increasingly negative over time. If σ is large, it is conceivable that b_2 will initially fall and then rise, although it appears unlikely on economic grounds. We conclude that the optimal b_2 is time-varying, unlike in the Grossman-Helpman model, and gradually moves to 1 from below, although conceivably non-monotonically.

[7] For this purpose, it is useful to work with the 'original' $\dot{n}_{S1} / n_{S1}$ and $\dot{n}_2 / n_2$ equations of SP1, which are derived by solving (A38) (differentiated with respect to t and then from (66) set equal to 0, and with $\phi = 0$), (A39), (A40), and the equation $\dot{n}_{S1} / n_{S1} + \dot{n}_2 / n_2 = L_r/a =$ $(1/a)$ – $\{1 + [\theta(\alpha_{S1}^{-1} - 1) - \theta_2(\alpha_2^{-1} - 1)] / \beta_2(\alpha_2^{-1} - 1)\} \rho / [\theta(\alpha_{S1}^{-1} - 1)]$, for $\dot{n}_{S1} / n_{S1}$, $\dot{n}_2 / n_2$, $\dot{q}_{S1} / q_{S1}$, and $\dot{q}_2 / q_2$.

Lastly, a precisely analogous characterization applies in respect of the evolution of b_{M1} in the preceding phase, in which decentralized implementation of SP6 is required. Thus, reflecting the optimal sequence of phases of structural change, *R&D* subsidization evolves from subsidizing input set *M*1, to set 2, to set *S*1, while intermediate-input purchase is optimally subsidized throughout.

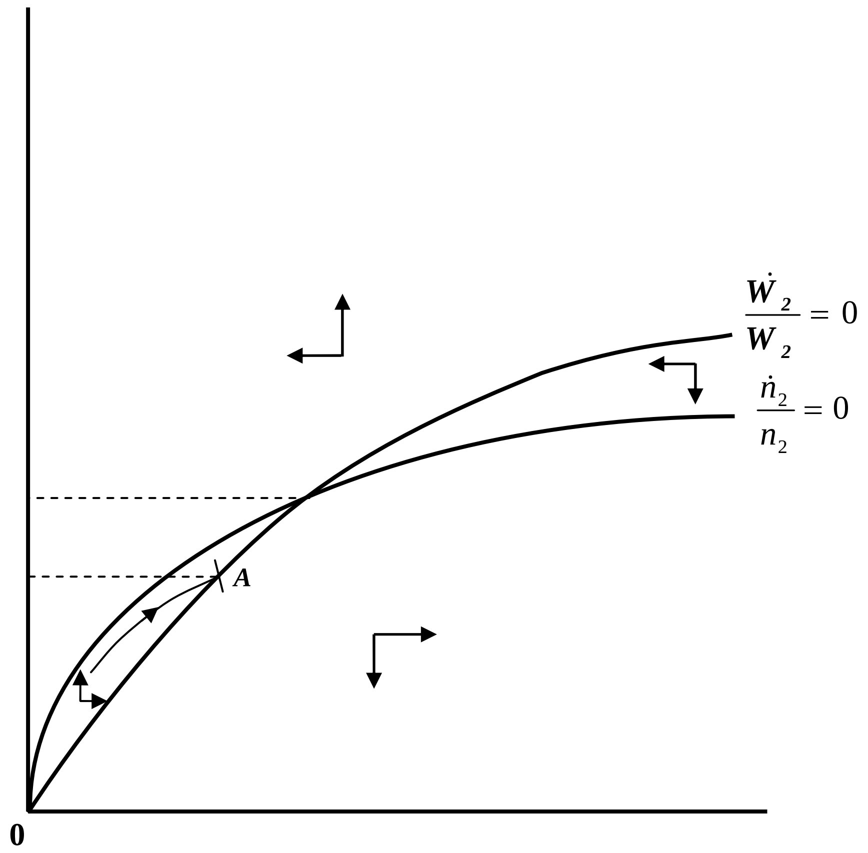

Figure 2

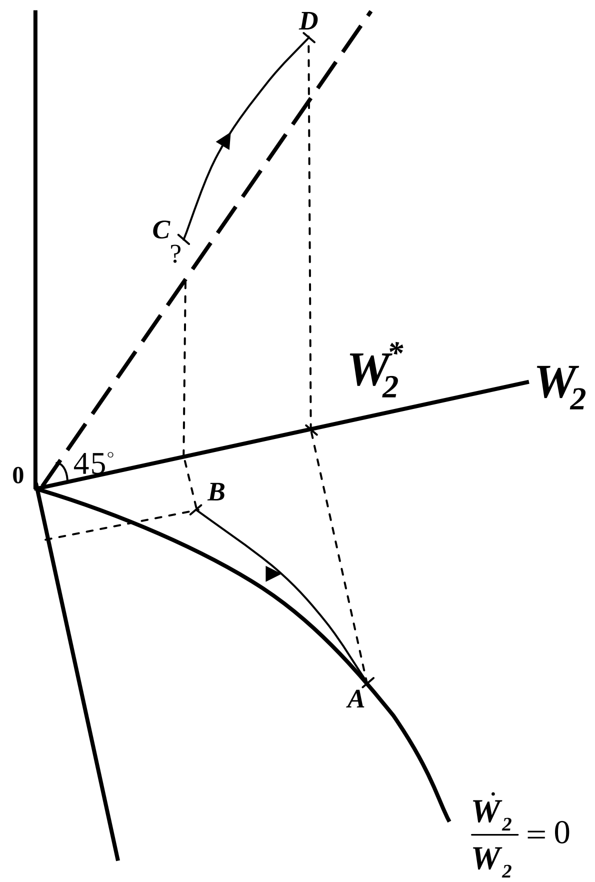

Figure 3